AB 1

1994

Let f be the function given by $f(x) = 3x^4 + x^3 - 21x^2$.

(a) Write an equation of the line tangent to the graph of f at the point $(2, -28)$.

(b) Find the absolute minimum value of f. Show the analysis that leads to your conclusion.

(c) Find the x-coordinate of each point of inflection on the graph of f. Show the analysis that leads to your conclusion.

(a) $f'(x) = 12x^3 + 3x^2 - 42x$

$f'(2) = 24$

$y + 28 = 24(x-2)$

or $y = 24x - 76$

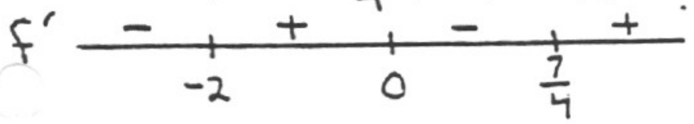
3 { 1: $f'(x)$
1: $f'(2)$
1: Equation of tangent

(b) $12x^3 + 3x^2 - 42x = 0$

$3x(4x^2 + x - 14) = 0$

$3x(4x-7)(x+2) = 0$

$x = 0,\ x = \frac{7}{4},\ x = -2$



min must be at −2 or $\frac{7}{4}$

$f(-2) = -44 \quad f(\frac{7}{4}) = -30.816$

Absolute min is −44

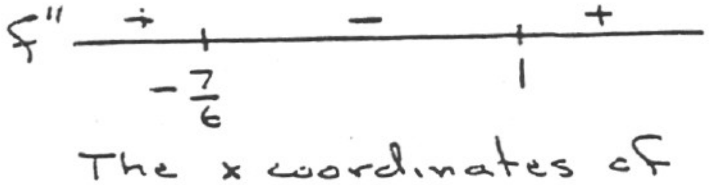
3 { 1: zeros of $f'(x)$
0/1: if f' quadratic
1: Analysis -- must indicate behavior of function over entire real line
1: Answer

(c) $f''(x) = 36x^2 + 6x - 42$

$= 6(6x^2 + x - 7)$

$= 6(6x+7)(x-1)$

Zeros at $x = -\frac{7}{6},\ x = 1$



The x coordinates of the points of inflection are $x = -\frac{7}{6}$ and $x = 1$

3 { 1: $f''(x)$
1: zeros of $f''(x)$
0/1: if f'' is linear
1: Answer {including analysis of sign of f'' or of concavity}

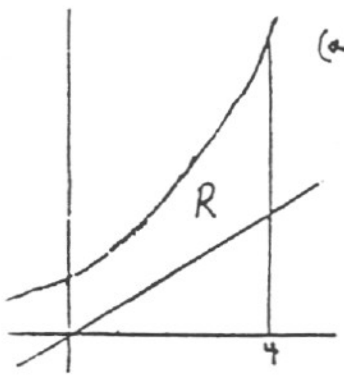
AB 2
BC 1
1994

Let R be the region enclosed by the graphs of $y = e^x$, $y = x$, and the lines $x = 0$ and $x = 4$.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the x-axis.

(c) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated when R is revolved about the y-axis.



(a)
$$\text{Area} = \int_0^4 e^x - x \, dx$$
$$= e^x - \frac{x^2}{2} \Big|_0^4$$
$$= \left(e^4 - \frac{16}{2}\right) - \left(e^0 - 0\right)$$
$$= e^4 - 9$$
$$\doteq 45.598$$

3
2: Integral
1: Limits
1: Integrand

1: Antidifferentation and evaluation
0/1 if integrand is not $e^x - x$ or $x - e^x$

(b)
$$V_x = \pi \int_0^4 (e^x)^2 - (x)^2 \, dx$$
$$= \pi \int_0^4 e^{2x} - x^2 \, dx$$
$$= \pi \left(\frac{1}{2}e^{2x} - \frac{x^3}{3}\right) \Big|_0^4$$
$$= \pi\left[\left(\frac{1}{2}e^8 - \frac{64}{3}\right) - \left(\frac{1}{2}e^0 - 0\right)\right]$$
$$= \pi\left(\frac{1}{2}e^8 - \frac{131}{6}\right)$$
$$\doteq 1468.646\pi \doteq 4613.886$$

4
3: Integral
1: π and correct limits
2: $(e^x)^2 - (x)^2$
<−1>: $(e^x - x)^2$
<−1>: $e^{x^2} - x^2$
<−1>: $x^2 - (e^x)^2$

1: Antidifferentiation and evaluation
0/1 if integrand is not $(e^x)^2 - (x)^2$ or $(x)^2 - (e^x)^2$

(c)
$$V_y = 2\pi \int_0^4 x(e^x - x) \, dx$$

- - - - - - - OR - - - - - - - -

$$V_y = \pi\left[\int_0^1 y^2 \, dy + \int_1^4 y^2 - (\ln y)^2 \, dy + \int_4^{e^4} 16 - (\ln y)^2 \, dy\right]$$

901.393

2
1: 2π and correct limits
1: Integrand

Note: $2\pi \int_0^4 x(e^x - x) \, dx$ in (b) and $\pi \int_0^4 (e^x)^2 - (x)^2 \, dx$ in (c) earns 0/4 in (b) and 2/2 in (c)

- - - - - - - OR - - - - - - - -

2
1: $\pi\left[\int_0^1 f(y) + \int_1^4 g(y) + \int_4^{e^4} h(y\right.$
1: All three integrands corre

Note: 0/2 if any limits incorrect

Board Note AB2 / BC1

#

Student who believes $y = e^x$ meets $y = x$

(a) $\int_0^k e^x - x \, dx + \int_k^4 x - e^x \, dx$, 0/2 Integral

Eligible for third point whether k is a number or parameter.

(b) $\pi \left[\int_0^k e^{2x} - x^2 \, dx + \int_k^4 x^2 - e^{2x} \, dx \right]$

Scores 1/3 for integral, and is eligible for fourth point.

(c) $2\pi \left[\int_0^k x(e^x - x) \, dx + \int_k^4 x(x - e^x) \, dx \right]$

Scores 1/2.

AB3
1.94

Consider the curve defined by $x^2 + xy + y^2 = 27$.

(a) Write an expression for the slope of the curve at any point (x, y).

(b) Determine whether the lines tangent to the curve at the x-intercepts of the curve are parallel. Show the analysis that leads to your conclusion.

(c) Find the points on the curve where the lines tangent to the curve are vertical.

(a) $2x + xy' + y + 2yy' = 0$ $y' = \dfrac{-2x-y}{x+2y}$	$3 \begin{cases} 2\text{: Implicit differentiation} \\ \quad \langle -1 \rangle \text{ each incorrect derivative of a term} \\ 1\text{: Solves for } y' \end{cases}$ (0/1 if only one y')
(b) If $y=0$, $x^2=27$ $x = \pm 3\sqrt{3}$ at $x=3\sqrt{3}$, $y' = \dfrac{-2\cdot 3\sqrt{3}}{3\sqrt{3}} = -2$ at $x=-3\sqrt{3}$, $y' = \dfrac{2\cdot 3\sqrt{3}}{-3\sqrt{3}} = -2$ Tangent lines at x-intercepts are parallel	$3 \begin{cases} 1\text{: Uses } y=0 \\ 1\text{: Finds } y' \text{ at x-intercepts} \\ 1\text{: Analysis and conclusion} \end{cases}$ NOTE: For solution at y-intercepts, eligible for 2nd and 3rd points
(c) y' undefined if $x+2y=0$ $(-2y)^2 + (-2y)y + y^2 = 27$ $3y^2 = 27$ $y = \pm 3$ Points are $(-6,3)$ and $(6,-3)$	$3 \begin{cases} 1\text{: Sets } x+2y=0 \\ 1\text{: Solves for } y \text{ (or } x\text{) values} \\ 1\text{: Gives both points} \end{cases}$ NOTE: max 1/3 for horizontal tangents

AB 4

1 94

A particle moves along the x-axis so that at any time $t > 0$ its velocity is given by $v(t) = t \ln t - t$. At time $t = 1$, the position of the particle is $x(1) = 6$.

(a) Write an expression for the acceleration of the particle.

(b) For what values of t is the particle moving to the right?

(c) What is the minimum velocity of the particle? Show the analysis that leads to your conclusion.

(d) Write an expression for the position $x(t)$ of the particle.

(a) $a(t) = v'(t) = \ln t + t \cdot \frac{1}{t} - 1 = \ln t$

1: $v'(t)$

(b) $v(t) = t\ln t - t > 0$

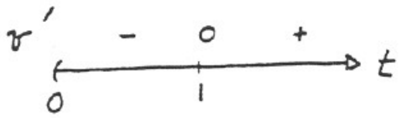
$t(\ln t - 1) > 0$

$t > e$

2 $\begin{cases} 1\text{: sets } t\ln t - t > 0 \\ 1\text{: answer} \end{cases}$

(c) $v'(t) = \ln t = 0$

$t = 1$



minimum velocity is $v(1) = -1$

3 $\begin{cases} 1\text{: solves } v'(t) = 0 \\ 1\text{: analysis} \\ 1\text{: answer} \end{cases}$

(max 1/3 if student's v' does not involve ln)

(d) $\int t\ln t - t \; dt$

$= \left(\frac{1}{2}t^2\ln t - \frac{1}{4}t^2\right) - \frac{1}{2}t^2 + C$

$= \frac{1}{2}t^2\ln t - \frac{3}{4}t^2 + C$;

$6 = x(1) = 0 - \frac{3}{4} + C$

$C = \frac{27}{4}$;

$x(t) = \frac{1}{2}t^2\ln t - \frac{3}{4}t^2 + \frac{27}{4}$

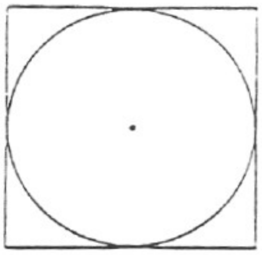
3 $\begin{cases} 2\text{: an antiderivative of } t\ln t - t \\ \quad \langle -2 \rangle\text{: error in } \int\text{-by-parts} \\ \quad \langle -1 \rangle\text{: any other error} \\ 1\text{: solves for } C \end{cases}$

(0/3 if not antidifferentiating $t\ln t - t$)

AB 5

C 2

1994

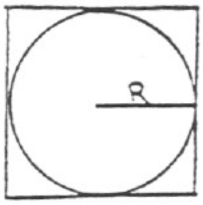


A circle is inscribed in a square as shown in the figure above. The circumference of the circle is increasing at a constant rate of 6 inches per second. As the circle expands, the square expands to maintain the condition of tangency. (Note: A circle with radius r has circumference $C = 2\pi r$ and area $A = \pi r^2$.)

(a) Find the rate at which the perimeter of the square is increasing. Indicate units of measure.

(b) At the instant when the area of the circle is 25π square inches, find the rate of increase in the area enclosed between the circle and the square. Indicate units of measure.

(a)



$$P = 8R$$

$$\frac{dP}{dt} = 8\frac{dR}{dt}$$

$C = 2\pi r$

$\cancel{2}\frac{dc}{dt} = 2\pi\frac{dr}{dt}$

$$6 = \frac{dC}{dt} = 2\pi\frac{dR}{dt}$$

$$\frac{dR}{dt} = \frac{3}{\pi}\ ; \quad \frac{dP}{dt} = \frac{24}{\pi} \text{ inches/second}$$

$$\doteq 7.639 \text{ inches/second}$$

4

- 1: Express perimeter of square in terms of circle data.
- 1: Differentiates to find $\frac{dP}{dt}$.
- 1: $\frac{dC}{dt} = 6$
- 1: Solves for $\frac{dP}{dt}$
  - 0/1 if no use of $\frac{dC}{dt}$.

(b) $\text{Area} = 4R^2 - \pi R^2$

$$\frac{d(\text{Area})}{dt} = 8R\frac{dR}{dt} - 2\pi R\frac{dR}{dt}$$

$$= (4-\pi)\,2R\frac{dR}{dt}$$

$$\text{Area of circle} = 25\pi = \pi R^2$$

$$R = 5$$

$$\frac{d(\text{Area})}{dt} = \frac{120}{\pi} - 30 \text{ inches}^2\text{/second}$$

$$= (4-\pi)\frac{30}{\pi} \text{ inches}^2\text{/second}$$

$$\doteq 8.197 \text{ inches}^2\text{/second}$$

4

- 1: Area of region between square and circle.
- 1: Derivative of area with respect to t.
  - 1/1 : For derivatives of areas of square and circle separately.
  - 0/1 : if area of either square or circle is linear.
- 1: Uses R = 5 in derivative
- 1: Answer
  - 0/1 : if chain rule error

- - - - - - - - - - - - -

units: inches/second in (a)

square inches/second in (b)

1

- 1: Units are correct for answers in both (a) and (b).

AB 6

1994

Let $F(x) = \int_0^x \sin(t^2)\,dt$ for $0 \le x \le 3$.

(a) Use the trapezoidal rule with four equal subdivisions of the closed interval [0, 1] to approximate $F(1)$.

(b) On what intervals is F increasing?

(c) If the average rate of change of F on the closed interval [1, 3] is k, find $\int_1^3 \sin(t^2)\,dt$ in terms of k

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(a) $F(1) = \int_0^1 \sin(t^2)\,dt$

$$\approx \frac{(1-0)}{4}\cdot\frac{1}{2}\cdot\left[\begin{array}{c}\sin 0^2 + 2\sin\left(\frac{1}{4}\right)^2 + 2\sin\left(\frac{1}{2}\right)^2 \\ + 2\sin\left(\frac{3}{4}\right)^2 + \sin 1^2\end{array}\right]$$

≈ 0.316

3

1: $\int_0^1 \sin(t^2)\,dt$ or $x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = \frac{3}{4}, x_4 = 1$

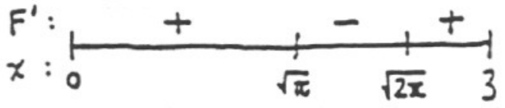
1: substitutes $x = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ into $\sin(x^2)$

1: Answer: uses $\sin(x^2)$-values in trapezoidal rule formula on $[0,1]$

(b) $F'(x) = \sin(x^2)$

$F'(x) = 0$ when $x^2 = 0, \pi, 2\pi, \ldots$

$x = 0, \sqrt{\pi}, \sqrt{2\pi}$



F is increasing on $[0, \sqrt{\pi}]$ and on $[\sqrt{2\pi}, 3]$

4

1: $F'(x) = \sin(x^2)$

1: Finds zeros of F' (& eliminates extraneous zeros,

1: Analyzes sign of F' within $[0,3]$

1: Indicates F increases where student's F' is positive

(c) $k = \dfrac{F(3) - F(1)}{2} = \dfrac{\int_1^3 \sin(t^2)\,dt}{2}$

$\int_1^3 \sin(t^2)\,dt = 2k$

2

1: $k = \dfrac{F(3) - F(1)}{2}$

1: Answer